Another View on Massless Matter-Gravity Fields

in Two Dimensions*

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Abstract

Conventional quantization of two-dimensional diffeomorphism and Weyl invariant theories sacrifices the latter symmetry to anomalies, while maintaining the former. When alternatively Weyl invariance is preserved by abandoning diffeomorphism invariance, we find that some invariance against coordinate redefinition remains: one can still perform at will transformations possessing a constant Jacobian. The alternate approach enjoys as much "gauge" symmetry as the conventional formulation.

A.   The theory of a massless scalar field ϕ, interacting with 2-dimensional gravity that is governed solely by a metric tensor $g_{\mu\nu}$ has a conventional description: functionally integrating ϕ produces an effective action Γ^P, a functional of $g^{\mu\nu}$, which has been given by Polyakov as [1]

$$\Gamma^P(g^{\mu\nu}) = \frac{1}{96\pi} \int d^2x\, d^2y \sqrt{-g(x)}\, R(x) K^{-1}(x,y) \sqrt{-g(y)} R(y) \tag{1}$$

Here R is the scalar curvature and K^{-1} satisfies

$$-\frac{1}{\sqrt{-g(x)}} \frac{\partial}{\partial x^\mu} \sqrt{-g(x)}\, g^{\mu\nu}(x) \frac{\partial}{\partial x^\nu} K^{-1}(x,y) = \frac{1}{\sqrt{-g(x)}} \delta^2(x-y) \tag{2}$$

Eq. (1) results after definite choices are made to resolve ambiguities of local quantum field theory: it is required that Γ^P be diffeomorphism invariant and lead to the conventional trace (Weyl) anomaly. This translates into the conditions that the energy-momentum tensor

$$\Theta_{\mu\nu}^P = \frac{2}{\sqrt{-g}} \frac{\delta \Gamma^P}{\delta g^{\mu\nu}} \tag{3}$$

be covariantly conserved (diffeomorphism invariance of Γ^P),

$$D_\mu \left(g^{\mu\nu} \Theta_{\nu\alpha}^P \right) = 0 \tag{4}$$

and possess a non-vanishing trace (Weyl anomaly).

$$g^{\mu\nu} \Theta_{\mu\nu}^P = \frac{1}{24\pi} R \tag{5}$$

Equations (3)–(5) can be integrated to give (1); also from (1) and (3) one finds that[1]

$$\Theta_{\mu\nu}^P = -\frac{1}{48\pi} \left(\partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi \right)$$
$$-\frac{1}{24\pi} \left(D_\mu D_\nu \Phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} D_\alpha D_\beta \Phi \right) + \frac{1}{48\pi} g_{\mu\nu} R \tag{6}$$

where Φ is the solution to

$$g^{\alpha\beta} D_\alpha D_\beta \Phi = \frac{1}{\sqrt{-g}} \partial_\alpha \sqrt{-g}\, g^{\alpha\beta} \partial_\beta \Phi = R \tag{7}$$

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[1]When the gravity field $g_{\mu\nu}$ is viewed as externally prescribed, $\Theta_{\mu\nu}^P$ is the vacuum matrix element of the operator energy-momentum tensor for the quantum field ϕ. Eq. (6) has been derived by M. Bos [2], not by varying the Polyakov action [1], but by direct computation of the relevant expectation value.

One easily verifies that (6) obeys (4) and (5). Notice that the traceless part of $\Theta^P_{\mu\nu}$ satisfies

$$D_\mu\left(g^{\mu\nu}\Theta^P_{\nu\alpha}\Big|_{\text{traceless}}\right) = \frac{1}{48\pi}\partial_\alpha R \tag{8}$$

It is well known that one can make alternative choices when defining relevant quantities. In particular, one can abandon diffeomorphism invariance and obtain an alternate effective action Γ, which is Weyl invariant because it is a functional solely of the Weyl invariant combination[2]

$$\gamma^{\mu\nu} \equiv \sqrt{-g}\,g^{\mu\nu} \tag{9}$$

This ensures vanishing trace for the modified energy momentum tensor.

$$\Theta_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\Gamma}{\delta g^{\mu\nu}} = 2\frac{\delta\Gamma}{\delta\gamma^{\mu\nu}} - \gamma_{\mu\nu}\gamma^{\alpha\beta}\frac{\delta\Gamma}{\delta\gamma^{\alpha\beta}} \tag{10}$$

$$\gamma^{\mu\nu}\Theta_{\mu\nu} = 0 \tag{11}$$

Here $\gamma_{\mu\nu}$ is the matrix inverse to $\gamma^{\mu\nu}$,

$$\gamma_{\mu\nu} = g_{\mu\nu}/\sqrt{-g} \tag{12}$$

and $\det\gamma_{\mu\nu} = \det\gamma^{\mu\nu} = -1$.

In this Letter we study more closely the response of Γ to diffeomorphism transformations when Weyl symmetry is preserved. We find that diffeomorphism invariance is not lost completely; rather it is reduced: Γ remains invariant against transformations that possess a constant (unit) Jacobian — we call this S-diffeomorphism invariance.[3] In the absence of diffeomorphism invariance, $\Theta_{\mu\nu}$ is no longer covariantly conserved; nevertheless we shall show that S-diffeomorphism invariance restricts the divergence of $\Theta_{\mu\nu}$ [essentially to the form given in (8)]. We shall argue that our alternative evaluation follows the intrinsic structures of the theory more closely than the conventional approach.

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[2]See Ref. [3]. A point of view that provides another alternative to Polyakov's approach has recently appeared in Ref. [4].

[3]S-diffeomorphisms preserve local area $\sqrt{-g}\,d^2x$ on spaces where $\sqrt{-g}$ is constant. (I thank W. Taylor and B. Zwiebach for discussions on this.)

B.   Before presenting our argument, we define notation and record some formulas. The 2-dimensional Euler density is a total derivative.

$$\sqrt{-g}R = \partial_\mu R^\mu \, , \qquad R = D_\mu \left(R^\mu / \sqrt{-g} \right) \tag{13}$$

But R^μ cannot be presented explicitly and locally in terms of the metric tensor and its derivatives as a whole; rather it is necessary to parametize $g_{\mu\nu} = \sqrt{-g}\gamma_{\mu\nu}$. We define

$$\sqrt{-g} = e^\sigma \tag{14a}$$

and parametize the light-cone $[(\pm) \equiv \frac{1}{\sqrt{2}}(0 \pm 1)]$ components of $\gamma_{\mu\nu}$ as

$$\gamma_{++} = -\gamma^{--} = e^\alpha \mathrm{sinh}\beta$$

$$\gamma_{--} = -\gamma^{++} = e^{-\alpha}\mathrm{sinh}\beta$$

$$\gamma_{+-} = \gamma_{-+} = \gamma^{+-} = \gamma^{-+} = \mathrm{cosh}\beta \tag{14b}$$

Then the formula for R^μ reads

$$R^\mu = \gamma^{\mu\nu}\partial_\nu\sigma + \partial_\nu\gamma^{\mu\nu} - \epsilon^{\mu\nu}(\mathrm{cosh}\beta - 1)\partial_\nu\alpha \tag{15}$$

where the explicit parametrization (14) is needed to present the last term in (15).[4] (Here $\epsilon^{\mu\nu}$ is the anti-symmetric numerical quantity, normalized by $\epsilon^{01} = 1$.)

Even though the last contribution in (15) to R^μ is not expressible in terms of $g_{\mu\nu}$ or $\gamma_{\mu\nu}$, its arbitrary variation satisfies a formula involving only $\gamma_{\mu\nu}$.

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[4]This is analogous to what happens with a Chern-Simons term. Upon performing a gauge transformation with a gauge function U, the Chern-Simons term changes by a total derivative. However, direct evaluation of the gauge response includes the expression $\omega = \frac{1}{24\pi^2}\mathrm{tr}\epsilon^{\alpha\beta\gamma}U^{-1}\partial_\alpha U U^{-1}\partial_\beta U^{-1}\partial_\gamma U$, which can be recognized as a total derivative only after U is explicitly parametrized. For example, in SU(2) $U = \exp\theta$, $\theta = \theta^a\sigma^a/2i$, and $\omega = \partial_\alpha\omega^\alpha$ where $\omega^\alpha = \frac{1}{4\pi^2}\mathrm{tr}\epsilon^{\alpha\beta\gamma}\theta\partial_\beta\theta\partial_\gamma\theta\left(\frac{|\theta|-\mathrm{sin}|\theta|}{|\theta|^3}\right)$ with $|\theta| \equiv \sqrt{\theta^a\theta^a}$; see Jackiw in [5].

$$\delta\Big[\epsilon^{\mu\nu}(\cosh\beta-1)\partial_\nu\alpha\Big] - \partial_\nu\Big[\epsilon^{\mu\nu}(\cosh\beta-1)\delta\alpha\Big] = \frac{-1}{2}\gamma^{\mu\nu}\Big(\partial_\alpha\gamma_{\nu\beta}+\partial_\beta\gamma_{\nu\alpha}-\partial_\nu\gamma_{\alpha\beta}\Big)\delta\gamma^{\alpha\beta} \quad (16)$$

Note that the right side equals $-\gamma^\mu_{\alpha\beta}\delta\gamma^{\alpha\beta}$, where $\gamma^\mu_{\alpha\beta}$ is the Christoffel connection when the metric tensor is $\gamma_{\mu\nu} : \gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta}\Big|_{g_{\mu\nu}=\gamma_{\mu\nu}}$

While the covariant divergence of $R^\mu/\sqrt{-g}$ is the scalar curvature, see (13), $R^\mu/\sqrt{-g}$ does not transform as a vector under coordinate redefinition. Rather for an infinitesimal diffeomorphism generated by f^μ

$$\delta_D x^\mu = -f^\mu(x) \quad (17)$$

one verifies that

$$\delta_D(R^\mu/\sqrt{-g}) = L_f(R^\mu/\sqrt{-g}) + \frac{1}{\sqrt{-g}}\epsilon^{\mu\nu}\partial_\nu\Delta_f \quad (18a)$$

where L_f in the Lie derivative with respect to f^μ, and

$$\Delta_f \equiv (\partial_+ - e^\alpha\tanh\frac{\beta}{2}\partial_-)f^+ - (\partial_- - e^{-\alpha}\tanh\frac{\beta}{2}\partial_+)f^- \quad (18b)$$

This non-tensorial transformation rule nevertheless ensures a scalar transformation law for $D_\mu(R^\mu/\sqrt{-g})$. Consequently, a world scalar action may be constructed by coupling vectorially R^μ to a scalar field Ψ, $I_V = \int d^2x\, R^\mu\partial_\mu\Psi$; invariance is verified from (13) after partial integration. An axial coupling also produces a world scalar action, $I_A = \int \frac{d^2x}{\sqrt{-g}}R_\mu\epsilon^{\mu\nu}\partial_\nu\Psi$, provided Ψ satisfies $g^{\mu\nu}D_\mu D_\nu\Psi = 0$; this follows from (18).

Finally we remark that the last term in (15) naturally defines a 1-form $a \equiv (\cosh\beta-1)d\alpha$ and the 2-form $\omega = da = \sinh\beta d\beta d\alpha$. These are recognized as the canonical 1- form and the symplectic 2-form, respectively, for SL $(2,R)$. Indeed ω also equals $\frac{1}{2}\epsilon_{abc}\xi^a d\xi^b d\xi^c$, where ξ^a is a three vector on a hyperboloid = SL $(2,R)/U(1) : (\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 = -1$. Effectively, ω is the Kirillov-Kostant 2-form on SL $(2,R)$.[5]

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[5]I thank V. P. Nair for pointing this out.

C. The Lagrange density for our theory reads

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \frac{1}{2}\gamma^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{19}$$

Equivalently, a first-order expression may be given,

$$\tilde{\mathcal{L}} = \Pi\dot{\phi} - u\mathcal{E} - vP \tag{20}$$

where $\mathcal{E}$ and $\mathcal{P}$ are the free-field energy and momentum densities

$$\mathcal{E} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\phi')^2 \tag{21a}$$

$$\mathcal{P} = -\phi'\Pi \tag{21b}$$

Here dot and dash signify time ($x^0 \equiv t$) and space ($x^1 \equiv x$) differentiation, respectively. The gravitational variables enter as Lagrange multipliers, in $\tilde{\mathcal{L}}$

$$u = \frac{1}{\sqrt{-g}\,g^{00}} = \frac{1}{\gamma^{00}}$$
$$v = \frac{g^{01}}{g^{00}} = \frac{\gamma^{01}}{\gamma^{00}} \tag{22}$$

enforce vanishing $\mathcal{E}$ and $\mathcal{P}$. It is seen that only two of the three independent components in $g_{\mu\nu}$ are present: $\sigma = \ln\sqrt{-g}$ does not occur in $\mathcal{L}$ or $\tilde{\mathcal{L}}$, which depend only on $\gamma^{\mu\nu}$ — this is of course a manifestation of Weyl invariance.

In spite of the absence of σ in the classical theory, Polyakov's quantum effective action (1) carries a σ-dependence. The breaking of Weyl symmetry arises when one evaluates the functional determinant that leads to the effective action; $viz.$ $-\frac{1}{2}\ln\det K$, where K is the kernel present in the classical action.

$$K(x,y) = -\frac{\partial}{\partial x^\mu}\gamma^{\mu\nu}(x)\frac{\partial}{\partial x^\nu}\delta^2(x - y) \tag{23}$$

Formally the determinant is given by the product of K's eigenvalues, $\det K = \Pi_\lambda\lambda$, but it still remains to formulate the eigenvalue problem. The diffeomorphism invariant definition recognizes that K is a density, so eigenvalues are defined by

$$\int K\Psi_\lambda^P = \sqrt{-g}\lambda\Psi_\lambda^P$$

$$-g^{\alpha\beta}D_\alpha D_\beta\Psi_\lambda^P = \lambda\Psi_\lambda^P \tag{24a}$$

and the inner product involves an invariant measure

$$\langle\lambda_1 \mid \lambda_2\rangle^P = \int \sqrt{-g}\Psi_{\lambda_1}^{P*}\Psi_{\lambda_2}^P \tag{24b}$$

In this way $\sigma = \ln\sqrt{-g}$ enters the calculation.

However, one may say that it is peculiar to introduce into the determination of eigenvalues a variable that is not otherwise present in the problem. (Below we shall also argue that it is unnatural to insist on diffeomorphism invariance.)

As an alternative to (24) one may define eigenvalues without inserting σ,

$$\int K\Psi_\lambda = \lambda\Psi_\lambda \tag{25a}$$

and use a σ-independent inner product.

$$\langle\lambda_1 \mid \lambda_2\rangle = \int \Psi_{\lambda_1}^*\Psi_{\lambda_2} \tag{25b}$$

It follows that the effective action will be as in (1), with σ set to zero.

$$\Gamma(\gamma^{\mu\nu}) = \frac{1}{96\pi}\int d^2x d^2y \mathcal{R}(x)K^{-1}(x,y)\mathcal{R}(y) \tag{26}$$

Here $\mathcal{R}$ is the scalar curvature computed with $\gamma^{\mu\nu}(\gamma_{\mu\nu})$ as the contravariant (covariant) metric tensor. From (13) – (15) we have

$$\mathcal{R} = \partial_\mu \mathcal{R}^\mu \tag{27}$$

$$\mathcal{R}^\mu = \partial_\nu\gamma^{\mu\nu} - \epsilon^{\mu\nu}(\cosh\beta - 1)\partial_\nu\alpha \tag{28}$$

Evidently Γ is a functional solely of $\gamma^{\mu\nu}$; since it does not depend on σ it is Weyl invariant, leading to a traceless energy-momentum tensor as in (11).

Of course the definitions (25) do not respect diffeomorphism invariance; however they are invariant against S-diffeomorphisms. Consequently Γ also is S-diffeomorphism invariant.

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With the help of (15), (26) and (28) we can exhibit the relation between Γ^P and Γ. Using (13) and (15) to evaluate (1), and integrating by parts the terms involving σ to remove the non-local kernel K^{-1}, leaves

$$\Gamma^P(g^{\mu\nu}) = \frac{1}{96\pi} \int d^2x \partial_\mu \sigma \gamma^{\mu\nu} \partial_\nu \sigma + \frac{1}{48\pi} \int d^2x \partial_\mu \sigma \mathcal{R}^\mu + \Gamma(\gamma^{\mu\nu}) \tag{29}$$

Thus the diffeomorphism invariance restoring terms, present in Γ^P, add to Γ a local expression, which is a quadratic polynomial in σ. The locality of $\Gamma^P - \Gamma$ highlights its arbitrariness, but Γ has the advantage of not involving quantities extraneous to the problem. [Formula (29) may also be presented as $\Gamma^P(g^{\mu\nu}) = \frac{1}{96\pi} \int (\sigma - K^{-1}\mathcal{R}) K (\sigma - K^{-1}\mathcal{R})$.]

Infinitesimal coordinate transformations make use of two arbitrary functions f^μ, see (17). S-diffeomorphisms possess unit Jacobian, so infinitesimally $\partial_\mu f^\mu = 0$; consequently only one function survives.

$$\delta_{SD} x^\mu = \epsilon^{\mu\nu} \partial_\nu f(x) \tag{30}$$

Since Weyl transformations

$$g_{\mu\nu} \to e^W g_{\mu\nu} \tag{31}$$

also make use of a single function, replacing diffeomorphism invariance, involving two arbitrary functions f^μ, by Weyl and S-diffeomorphism invariance still leaves two arbitrary functions, f and W. Indeed, similar to diffeomorphism invariance, the combination of Weyl and S-diffeomorphism invariance can be used to reduce a generic metric tensor, containing three functions, to a single arbitrary function.

In particular by using their respective symmetries, we can bring $\Gamma^P(g^{\mu\nu})$ and $\Gamma(\gamma^{\mu\nu})$ into equality. Diffeomorphism invariance allows placing $g_{\mu\nu}$ into the light-cone gauge, where $g_{--} = 0$, $g_{+-} = 1$ and g_{++} is the arbitrary function h_{++} [1]. Correspondingly, with S-diffeomorphism invariance we can set to zero the $(--)$ component in $\gamma_{\mu\nu}$ and the $(+-)$ component to unity. This is achieved by passing from the original variables $\{x^\mu\}$ and metric function $\gamma_{\mu\nu}(x)$ to a new quantities $\{\tilde{x}^\mu\}$ and $\tilde{\gamma}_{\mu\nu}(\tilde{x})$, where

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$$\frac{\partial x^+}{\partial \tilde{x}^-} = -\frac{\gamma_{--}}{\gamma_{+-} \pm 1} \tag{32a}$$

$$\frac{\partial x^+}{\partial \tilde{x}^+} = \frac{-\gamma_{--}}{\gamma_{+-} \pm 1} \frac{\partial x^-}{\partial \tilde{x}^-} + \frac{c}{\partial x^-/\partial \tilde{x}^-}$$

Either sign may be taken in $\gamma_{+-} \pm 1$ and $c^2 = 1$. One then finds

$$\tilde{\gamma}_{--} = 0, \qquad \tilde{\gamma}_{+-} = 1$$

$$\tilde{\gamma}_{++}(\tilde{x}) = \frac{\gamma_{++}(x)}{(\partial x^-/\partial \tilde{x}^-)^2} + 2c \frac{\partial x^-/\partial \tilde{x}^+}{\partial x^-/\partial \tilde{x}^-} \tag{32b}$$

Upon identification of $\tilde{\gamma}_{++}$ with h_{++}, $\Gamma^P = \Gamma$ in the selected gauge.

Under an infinitessimal diffeomorphism

$$\delta_D \Gamma = \int d^2 x \sqrt{-g} f^\alpha D_\mu \Theta^\mu{}_\alpha \tag{33}$$

so it follows from (30) that for S-diffeomorphisms

$$\delta_{SD} \Gamma = \int d^2 x f \epsilon^{\alpha\beta} \partial_\beta \left(\sqrt{-g} D_\mu \Theta^\mu{}_\alpha \right) \tag{34}$$

and invariance is equivalent to vanishing of the integrand. But $\sqrt{-g} D_\mu \Theta^\mu{}_\alpha = \partial_\mu(\sqrt{-g} g^{\mu\nu} \Theta_{\nu\alpha}) + \frac{\sqrt{-g}}{2} \partial_\alpha g^{\mu\nu} \Theta_{\mu\nu}$, which for traceless $\Theta_{\mu\nu}$ may be written as $\partial_\mu(\gamma^{\mu\nu} \Theta_{\nu\alpha}) + \frac{1}{2} \partial_\alpha \gamma^{\mu\nu} \Theta_{\mu\nu} = d_\mu(\gamma^{\mu\nu} \Theta_{\nu\alpha})$, where d_μ is a covariant derivative constructed from $\gamma_{\mu\nu}$. Consequently, the restriction given by S-diffeomorphism invariance can be presented in a S-diffeomorphism invariant way as

$$d_\mu d_\nu \left(\epsilon^{\mu\alpha} \Theta_{\alpha\beta} \gamma^{\beta\nu} \right) = 0 \tag{35a}$$

This implies that

$$d_\mu \left(\gamma^{\mu\nu} \Theta_{\nu\alpha} \right) = \partial_\alpha \text{ (scalar)} \tag{35b}$$

which is the constraint on the divergence of the energy-momentum tensor mentioned earlier.

Computing $\Theta_{\mu\nu}$ from (29) gives

$$\Theta_{\mu\nu} = -\frac{1}{48\pi} \left(\partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} \gamma_{\mu\nu} \gamma^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi \right) - \frac{1}{24\pi} \left(d_\mu d_\nu \varphi - \frac{1}{2} \gamma_{\mu\nu} \gamma^{\alpha\beta} d_\alpha d_\beta \varphi \right) \tag{36}$$

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where φ satisfies [compare (6) and (7)]

$$\gamma^{\alpha\beta} d_\alpha d_\beta \varphi = \partial_\alpha \gamma^{\alpha\beta} \partial_\beta \varphi = \mathcal{R} \tag{37}$$

Clearly $\Theta_{\mu\nu}$ is traceless, and one readily verifies that

$$d_\mu \left(\gamma^{\mu\nu} \Theta_{\nu\alpha} \right) = \frac{1}{48\pi} \partial_\alpha \mathcal{R} \tag{38}$$

[compare (8)], which is consistent with (35).

Finally we remark that even under S-diffeomorphisms $\mathcal{R}^\mu$ does not transform as a vector. One finds from (15), (18) and (28)

$$\delta_{\mathrm{SD}} \mathcal{R}^\mu = L_f \mathcal{R}^\mu + \epsilon^{\mu\nu} \partial_\nu \Delta_f \tag{39}$$

where the vector field f^μ is now $-\epsilon^{\mu\nu} \partial_\nu f$; thus here $\Delta_f = 2\partial_+ \partial_- f - \tanh \frac{\beta}{2} \left(e^\alpha \partial_-^2 + e^{-\alpha} \partial_+^2 \right) f$.

Although selecting between Weyl and S-diffeomorphism invariance on the one hand or conventional diffeomorphism invariance on the other remains a matter of arbitrary choice, as is seen from the fact that the effective actions for the two options differ by local terms, the following observations should be made in favor of the former.

D. Up to now, the gravitational field $g_{\mu\nu}$ was a passive, background variable. Consider now the puzzles that arise when it is dynamical; *i.e.* $g_{\mu\nu}$ is varied. With a single Bose field, it is immediately established that the classical theory does not possess solutions. This is seen from the equation that follows upon varying $g_{\mu\nu}$ in $\mathcal{L}$,

$$\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi = 0 \tag{40}$$

which implies that $g_{\mu\nu} \propto \partial_\mu \phi \partial_\nu \phi$, so that g vanishes and $g^{\mu\nu}$ does not exist; alternatively ϕ must be constant and $g_{\mu\nu}$ undetermined. If there are N scalar fields, whereupon the effective action acquires the factor N, the above difficulty is avoided, because $g_{\mu\nu} \propto \sum_{i=1}^N \partial_\mu \phi^i \partial_\nu \phi^i$ need not be singular. Nevertheless (40) (with field bilinears replaced by sums over the N fields) requires the vanishing of positive quantities $\sum_{i=1}^N \left\{ \left(\gamma^{00} \dot\phi^i + \gamma^{01} \phi'^i \right)^2 + (\phi'^i)^2 \right\}$ and again only the trivial solution is allowed.

The quantum theory in Hamiltonian formulation also appears problematic, in that the constraints of vanishing $\mathcal{E}$ and $\mathcal{P}$ cannot be imposed on states. With one scalar field, the momentum constraint requiring that $\phi'\Pi$ acting on states vanish — this is the spatial diffeomorphism constraint — forces the state functional in the Schrödinger representation to have support only for constant fields ϕ. (Equivalently one observes that a spatial diffeomorphism invariant functional cannot be constructed from a single, x-dependent field.) With more than one field, this problem is absent [a diffeomorphism invariant functional can involve $\int dx \phi_1(x)\phi_2'(x)$] and the momentum constraint can be solved. However, an obstruction remains to solving the energy constraint, owing to the well-known Schwinger term (Virasoro anomaly) in the $[\mathcal{E}, \mathcal{P}]$ commutator, which gives a central extension that interferes with closure of constraints: classical first-class constraints become upon quantization second-class.

$$i\,[\mathcal{E}(x), \mathcal{E}(y)] = i\,[\mathcal{P}(x), \mathcal{P}(y)] = (\mathcal{P}(x) + \mathcal{P}(y))\delta'(x-y)) \tag{41a}$$

$$i\,[\mathcal{E}(x), \mathcal{P}(y)] = (\mathcal{E}(x) + \mathcal{E}(y))\,\delta'(x-y) - \frac{N}{12\pi}\delta'''(x-y) \tag{41b}$$

Note that all the above troubles, both in the classical theory and in the Dirac-quantized Hamiltonian theory, revolve around diffeomorphism invariance, not Weyl invariance. Indeed the same troubles persists for massive scalar fields, which are not Weyl invariant.

Thus when a quantum theory is constructed by a functional integral (not by Hamiltonian/Dirac quantization) it is natural that it should reflect problems with diffeomorphism invariance — reducing it to S-diffeomorphism invariance. Weyl invariance on the other hand could survive quantization.

REFERENCES

[1] A. Polyakov, *Mod. Phys. Lett.* **A2** (1987) 893; *Gauge Fields and Strings* (Harwood, New York, NY, 1987); see also M. Duff, *Nucl. Phys.* **B 125** (1977) 334.

[2] M. Bos, *Phys. Rev. D* **34** (1986) 3750.

[3] K. Fujikawa, V. Lindström, N. K. Nielsen, M Roček, and P. van Nieuwenhuizen, *Phys. Rev. D* **37** (1988) 391; M. Hatsuda, P. van Nieuwenhuizen, W. Troost, and A. van Proeyen, *Nucl. Phys.* **B 335** (1990) 166.

[4] T. Fujiwara, Y. Igarashi, J. Kubo and T. Tabei, *Phys. Lett.* **B 336** (1994) 157.

[5] S. Treiman, R. Jackiw, B. Zumino and E. Witten, *Current Algebra and Anomalies* (Princeton University Press/World Scientific, Princeton, NJ/Singapore, 1985).

ADDED NOTE: Many of the results presented here on S-diffeomorphism invariance have already appeared in D. Karakhanyan, R. Manvelyan and R. Mkrtchyan, *Phys. Lett.* **B 329** (1994) 185. Hence my paper is not for publication.